

Paul Langevin and the Sagnac effect (1921)

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Paul Langevin (CRAS of 07.11.1921 "On the theory of relativity and the experience of Mr. Sagnac") meets the concerns of some of his colleagues from the Academy of Sciences who wondered if the experience of Sagnac², does not invalidate special relativity (stipulating that one cannot detect the movement of an inertial system by performing internal experiences in the system)³, since this experience can detect rotation and secondly they wondered what relativity predicts for this experience.

Paul Langevin will propose a geometric resolution of the problem by using the relativistic metric. To show that the first order relativistic solution converges with the Newtonian solution, P. Langevin will assume that the tangential speed is very small compared to that of light ($1 \pm \omega^2 r^2/c^2 \approx 1$).

Langevin also points out that this experience, which has a result not null at first order is less discriminating for validating the different theories than that of Michelson which is an experience at second order.

We keep the principle of his method, but by generalizing it and by using polar coordinates more convenient for this problem than the Cartesian coordinates he uses in his demonstration.

The result at first order will be found by the same approximation as him, but deriving from the general result.

Minkowski's metric associated with the global frame (R_0), external to the instrument:

$$ds^2 = c^2 dt^2 - r^2 d\Phi^2 \quad [1]$$

Metric on the local frame (R_1) in ω angular velocity co-rotation associated with the instrument:

$$ds^2 = (c^2 - \omega^2 r^2) dt^2 - 2\omega r^2 dt \cdot d\varphi - r^2 \cdot (d\varphi)^2 \quad [2]$$

Because: $(\Phi) = (\varphi) + \omega \cdot t$

Note that the coordinate t is the same in [1] and [2]. In [1], t is the proper time of a static observer.

For a photon $ds^2 = 0$, therefore [2] becomes:

$$(c^2 - \omega^2 r^2) dt^2 - 2\omega r^2 dt \cdot d\varphi - r^2 d\varphi^2 = 0 \quad [3]$$

We consider [3] as a quadratic equation in dt . This gives for the 2 roots:

$$dt = (d\varphi) [(\omega r^2/c^2) \pm (r/c)] / (1 - \omega^2 r^2/c^2) \quad [4]$$

One of the roots is relative to the co-rotating photon and the other to the counter-rotating-photon.

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² Sagnac uses a rotating device which split a light beam in two light beams of opposite direction. A set of mirrors recombines the two beams to an interferometer for detecting motion.

³ A rotating system is not inertial.

Integration⁴ of φ , from 0 to 2π , where $A = \pi.r^2$ is the area of the disk and $L= 2\pi r$ is the perimeter of the disk, yields:

$$t = [(2\pi\omega r^2)/c^2 \pm L/c] / (1 - \omega^2 r^2/c^2) = [(2\omega A)/c^2 \pm L/c] / (1 - \omega^2 r^2/c^2) \quad [5]$$

t is the time coordinate of the metric, which is also the proper time of the "static external" observer in the R_0 frame. This is what an outside observer would see about the rotating instrument.

As the instrument (interferometer) is on the rotating platform (experience internal to the system), we have to take into account the proper time, τ , of the (virtual) observer, attached to the rotating frame, for the measurement. With $\varphi = \text{constant}$, [2] becomes:

$$d\tau^2 = ds^2/c^2 = (1 - \omega^2 r^2/c^2) dt^2 \quad [6]$$

By integrating $d\tau = dt (1 - \omega^2 r^2/c^2)^{1/2}$, we get:

$$\tau = t (1 - \omega^2 r^2/c^2)^{1/2} \quad [7]$$

Plugging in [5]

$$\tau = [(2\omega A)/c^2 \pm L/c] / (1 - \omega^2 r^2/c^2)^{1/2} \quad [8]$$

Therefore the difference $\Delta\tau$ between the travel time τ_1 of a co-rotating photon and τ_2 the travel time of a counter-rotating photon in the frame (R_1) measured on the rotating platform is:

$$|\tau_1 - \tau_2| = \Delta\tau = [(4\omega A)/c^2] / (1 - \omega^2 r^2/c^2)^{1/2}$$

This is what observes, (measured with a clock or by observing the interference fringes), an observer on the rotating instrument.

By neglecting $\omega^2 r^2/c^2 \ll 1$, we obtain the result given by Newtonian mechanics:

$$\Delta\tau_{1st} = [(4\omega A)/c^2]$$

This is the result of the calculation made by Paul Langevin who considered this first order approximation early on in his article which is identical to that of the Newtonian mechanics.

4 The integration constant can be ignored as it is eliminated in the operation computing the time difference.